

Trace anomaly and Casimir effect

M.R.Setare* and A.H.Rezaeian[†]

Department of Physics, Sharif University of Technology Tehran-Iran

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Abstract

The Casimir energy for scalar field of two parallel conductors in two dimensional domain wall background, with Dirichlet boundary conditions, is calculated by making use of general properties of renormalized stress tensor. We show that vacuum expectation values of stress tensor contain two terms which come from the boundary conditions and the gravitational background. In two dimensions the minimal coupling reduces to the conformal coupling and stress tensor can be obtained by the local and non-local contribution of the anomalous trace. This work shows that there exists a subtle and deep connection between Casimir effect and trace anomaly in curved space time.

*E-mail: Mreza@physics.sharif.ac.ir

[†]E-mail: Rezaeian@physics.sharif.ac.ir

1 Introduction

In the semiclassical approximation theory of quantum gravity we are involved with calculation the expectation value of energy momentum tensor in special vacuum, [1]. However the usual expression for the stress tensor includes singular products of the field operators for stress tensor. Renormalization theory of the stress tensor claims to solve this problem, but it must be mentioned that the usual scheme of renormalization includes complexity and somewhat ambiguity. For instance, there is no conceptual support for a local measure of energy momentum of some given state without any reference to any global structure. We know in this frame energy is source of gravity and we are not allowed to subtract any unwanted part of energy even though it is infinite. So to consider the back reaction effect of the quantum field on the gravitational field, we must find a more elaborate renormalization scheme in which the dynamics of gravitational field is a vital component. In original Casimir effect discovered in 1948 by H.B.G Casimir [2] we are concerned about force and energy, but we are not usually interested in dynamics of the gravitational field. Even in many cases in curved boundary problems, energy is not our main concern. Because of unphysical nature of boundary condition the energy diverges approaching to curved boundary [3]. The Casimir effect can be viewed as a polarization of vacuum by boundary conditions and external fields, such as gravitational field. In the present paper we are going to consider a simple example in which these two types of sources for vacuum polarization are present. There is several methods for calculating Casimir energy. For instance, we can mention mode summation, Green's function method [4], heat kernel method [5] along with appropriate regularization schemes such as point separation [6],[7] dimensional regularization [8], zeta function regularization [9]. But it must be remarked that practically all of the methods are successful only for boundary conditions with high symmetry in flat space time. In fact we don't have any general procedure for renormalizing stress tensor in gravitational background with arbitrary boundary.

In this paper the Casimir stress tensor for scalar field in two dimensional analog of domain wall space time for two parallel conductor plates with Dirichlet boundary conditions, is calculated. The Casimir stress tensor is obtained by imposing only general requirements which is discussed in section 2. We show direct relation between trace anomaly and Casimir effect, although we have been aware of role of anomalous trace in gravitational background such as Hawking effect [10].

2 General properties of stress tensor

In semiclassical framework for yielding a sensible theory of back reaction Wald [11] has developed an axiomatic approach. There one tries to obtain an expression for the renormalized $T_{\mu\nu}$ from the properties (axioms) which it must fulfill. The axioms for the renormalized energy momentum tensor are as follow:

- 1-For off-diagonal elements standard results should be obtained.
- 2-In Minkowski space time standard results should be obtained.
- 3-Expectation values of energy momentum are conserved.
- 4-Causality holds .
- 5-Energy momentum tensor contains no local curvature tensor depending on derivatives of the metric higher than second order.

Two prescriptions that satisfy the first four axioms can differ at most by a conserved local curvature term. Wald, [12], showed any prescription for renormalized $T_{\mu\nu}$ which is consistent with axioms 1-4 must yield the given trace up to the addition of the trace of conserved local curvature. It must be noted (that trace anomalies in stress-tensor, that is, the nonvanishing T_μ^μ for a conformally invariant field after renormalization) are originated from some quantum behavior [13]. In two dimensional space time one can show that a trace-free stress tensor can not be consistent with conservation and causality if particle creation occurs. A trace-free, conserved stress tensor in two dimensions must always remain zero if it is initially zero. One can show that the "Davies-Fulling-Unruh" [14] formula for the stress tensor of scalar field which yield an anomalous trace, $T_\mu^\mu = \frac{R}{24\pi}$, is the unique one which is consistent with the above axioms. In four dimensions, just as in two dimensions, a trace-free stress tensor which agrees with the formal expression for the matrix elements between orthogonal states can not be compatible with both conservation laws and causality. It must be noted that, as showed Wald[12], with Hadamard regularization in massless case axiom(5) can not be satisfied unless we introduce a new fundamental length scale for nature. Regarding all these axioms, thus, we are able to get an unambiguous prescription for calculation of stress tensor.

3 Vacuum expectation values of stress tensor

It has been shown in [15],[16] that the gravitational field of the vacuum domain wall with a source of the form

$$T_\mu^\nu(x) = \sigma\delta(x)\text{diag}(1, 0, 1, 1) \quad (1)$$

does not correspond to any exact static solution of the Einstein's equation (for domain wall solution of Einstein-scalar equation see[17]). However a static solution of the Einstein's equation representing a planar domain wall in an anisotropic background has been found [18]. This solution matches in the weak-field region to the linearized solution of Vilenkin. The energy momentum tensor of the background has the form

$$T_\nu^\mu = \frac{\alpha(\alpha - 2\gamma)k^2(1 + kx)^{2(\alpha+\gamma)}}{8\pi G}\text{diag}(1, \frac{-3\alpha}{2\gamma - \alpha}, 1, 1) \quad (2)$$

For the energy density of the background to be positive we must have $\gamma < \frac{\alpha}{2}$, then we have the following Ricci tensor

$$R_\nu^\mu = \alpha(\gamma - 2\alpha)k^2(1 + kx)^{2(\alpha+\gamma)}\text{diag}(1, \frac{3\gamma}{\gamma - 2\alpha}, 1, 1) \quad (3)$$

In this case we have the metric

$$ds^2 = (1 + kx)^{-2\alpha}(dt^2 - dy^2 - dz^2) - (1 + kx)^{-2(\alpha+\gamma+1)}dx^2 \quad (4)$$

where $\alpha, \gamma, k > 0$, This corresponds to a domain wall with surface tension $\sigma = \frac{\alpha k}{2\pi G} > 0$. The space time on the domain wall is flat, and energy momentum tensor of the wall is invariant with respect to Lorentz boost in the $y - z$ plane. The above metric representing a gravitational field which is homogeneous and isotropic in the $y - z$ plane and has reflection symmetry with respect to the wall. In fact(4) is the special form of the following metric

$$ds^2 = (1 + k|x|)^{-2\alpha}dt^2 - (1 + k|x|)^{-2(\alpha+\gamma+1)}dx^2 + (1 + k|x|)^{-2\beta}(dy^2 + dz^2) \quad (5)$$

The case $\beta = 0$ and $\gamma = \alpha$ corresponds to the flat Kanser metric in Taub coordinates [19]. In metric (4) we have $\alpha = \beta$. Now, just for the sake of simplicity, we consider two dimensions in which

$$d^2s = (1 + kx)^{-2\alpha}(d^2t - d^2X) \quad (6)$$

where

$$X = \frac{1 - (1 + kx)^{-\gamma}}{k\gamma} \quad (7)$$

Let's define

$$\Omega(x) = (1 + kx)^{-2\alpha} \quad (8)$$

and so we have

$$\frac{dx}{dX} = \Omega(x)^{k'} \quad k' = \frac{-(\gamma + 1)}{2\alpha} \quad (9)$$

From now on, our main goal is to determine for a general form of conserved energy momentum tensor, regarding trace anomaly for the metric (4). For the non -zero Christoffel symbols of the metric (4) we have in (t,X) coordinate;

$$\Gamma_{tt}^X = \Gamma_{tX}^t = \Gamma_{XX}^t = \Gamma_{XX}^X = \frac{\Omega^{k'-1}(x)}{2} \frac{d\Omega(x)}{dx} \quad (10)$$

Then the conservation equation takes the following form

$$\partial_X T_t^X + \Gamma_{tX}^t T_t^X - \Gamma_{tt}^X T_X^t = 0 \quad (11)$$

$$\partial_X T_X^X + \Gamma_{tX}^t T_X^X - \Gamma_{tX}^t T_t^t = 0 \quad (12)$$

in which,

$$T_X^t = -T_t^X \quad T_t^t = T_\beta^\beta - T_X^X \quad (13)$$

and T_β^β is anomalous trace in two dimension. Using the equations (10 – 12) it could be shown that

$$\frac{d(T_t^X \Omega(x)^{k'})}{dx} = 0 \quad (14)$$

and

$$\frac{d(\Omega^{k'}(x) T_X^X)}{dx} = (1/2 \Omega^{k'-1}(x)) \left(\frac{d\Omega(x)}{dx} \right) T_\beta^\beta \quad (15)$$

Then equation (14) leads to:

$$T_t^X = \alpha' \Omega^{-k'}(x) \quad (16)$$

where α' is the constant of integration. The solution of Eq.(15) might be written in the following form

$$T_X^X(x) = (H(x) + \eta) \Omega^{-k'}(x) \quad (17)$$

where

$$H(x) = 1/2 \int_l^r T_\beta^\beta(x') \Omega^{k'-1}(x') \frac{d\Omega(x')}{dx'} dx' \quad (18)$$

With l being an arbitrary scale of length and considering

$$T_\beta^\beta = \frac{R}{24\pi} = \frac{\alpha(\gamma - 2\alpha)k^2(1 + kx)^{2(\alpha+\gamma)}}{12\pi} \quad (19)$$

the function $H(x)$ produces the non-local contribution of the trace $T_\beta^\beta(x)$ to the energy momentum tensor. Finding l depends on the metric. For the metric (4) we choose $l = 0$ (to be definite we shall consider right half space of domain wall geometry) because we want to incorporate the non-local effect of trace anomaly in all space time, so we reach

$$H(x) = \frac{(2\alpha - \gamma)\alpha^2 k^2 ((kx + 1)^{2\alpha+3\gamma+1} - 1)}{12\pi(2\alpha + 3\gamma + 1)} \quad (20)$$

Using the equations (13), (16) and (17) it can be shown that energy momentum tensor takes the below form in (t, X) coordinates. So we have most general form of stress tensor field in our interesting background.

$$T^\mu{}_\nu(x) = \begin{pmatrix} T_\beta^\beta - \Omega(x)^{-k'} H(x) & 0 \\ 0 & \Omega(x)^{-k'} H(x) \end{pmatrix} + \Omega^{k'} \begin{pmatrix} -\eta & -\alpha' \\ \alpha' & \eta \end{pmatrix} \quad (21)$$

Now we are going to obtain two constants α' and η by imposing the second axiom of renormalization scheme. So when we put $k = 0$, we reach the special case of Minkowski space time. The type of boundary condition which we choose is Dirichlet $\phi(x_1) = \phi(x_2) = 0$. The standard Casimir stress in Minkowski space time is as follows

$$reg < 0 | T_{\mu\nu} | 0 > = \frac{-\pi}{24a^2} \delta_{\mu\nu} \quad (22)$$

where a is proper distance between the plates. For example, it could be obtained by mode summation using Abel-Plana formula [20], without inserting any cut off. Comparing to (21) we obtain

$$\eta = \frac{\pi}{24a^2} \quad \alpha' = 0 \quad (23)$$

Thus we have obtained the energy momentum tensor as direct sum of two terms; boundary term (second term) and term which presents the vacuum polarization in gravitational background in absence of boundaries (first term).

$$< T_\nu^\mu > = < T_\nu^{(g)\mu} > + < T_\nu^{(b)\mu} > \quad (24)$$

where $< T_\nu^{(g)\mu} >$ and $< T_\nu^{(b)\mu} >$ stand for gravitational and boundary parts respectively. It should be noted that trace anomaly has contribution in first term which comes from background not boundary effect. However it has contribution in total Casimir energy momentum tensor. As in four dimension previously has been shown [21] in the regions $x < x_1$ and $x > x_2$ the boundary part is zero and only gravitational polarization part is present, it is clear that the forces acting on plates are determined only by boundary part, when the effective pressure created by gravitational part is zero.

4 Conclusion

We have found the renormalized energy momentum tensor for scalar field with Dirichlet boundary conditions in domain wall background, only by making use of general properties of stress tensor. It is in close relation to what is done in [10], [22]. We propose that if we know the stress tensor for a given boundary in Minkowski space time, Casimir effect in gravitational background can be calculated. The result contains two parts, one comes from

boundary conditions and the other one comes from the effect of gravitational background over the vacuum of scalar field, this part carries the local and nonlocal contributions of anomalous trace in complete Casimir effect in curved background.

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References

- [1] N.D.Birrell and P.C.W Davies "Quantum fields in curved space". (Chambridge press,1982)
- [2] H.B.G.Casimir,proc.kon.ned.akad.wetensch 51,793(1948)
- [3] D.Deutsch and P.Candelas Phys.Rev.D20,3063(1979)
- [4] G.Plunien,B.Muller,W.Greiner,Phys.Rep.134,87(1986)
- [5] K.Bormann and Antonsen"The Casimir effect of curved spacetime"LANL preprint:hep-th/9608142.
- [6] S.M.Christensen,Phys.Rev.D14,2490(1976);17,946 1978
- [7] S.L.Adler ,J.Lieberman and Y.J.Ng Ann.Phy.(N.Y) 106,279(1977)
- [8] Deser,M.J.Duff and C.J.Isham Nucl.Phys B11 45 (1976)see also D.M.capper and M.J.Duff,Nuovo cimento 23A,173(1974);Phys.Lett.53A,361(1975)
- [9] S.W.Hawking,Commun.Math.Phys.55,133(1977)
- [10] S.M.Christensen and S.A.Fulling,Phys.Rev.D15,2088(1977)
- [11] R.M.Wald,Commun.Math.Phys.54,1(1977)
- [12] R.M.Wald,Phys.Rev.D17,1477,(1978)
- [13] S.Coleman and R.Jackiw,Ann.Phys.(NY)67,552(1971)
- [14] D.C.W.Davies,S.A.Fulling and W.G .Unruh,Phys.Rev D13,2720(1976) and also see A.Z.capri,M.Kobayashi and D.J.Lamb Class.Quant.Grav.13(1996)179-188
- [15] A.Vilenkin, Phys.Rev.D23,852(1981)
- [16] A.Vilenkin,Phys.Lett.133B,177(1983)
- [17] L.M.Widrow,Phys.Rev.D39,3571(1989)
- [18] R.Mansouri,Gravitation field of static domain wall preprint UWThPh-1989-12.
- [19] P.A.Amundsen and ö.GrØn,Phys.Rev.D27,1731(1983)
- [20] A.A.Saharian"The Generalized Abel-Plana formula application to bessel function"LANL preprint:hep-th/0002239
- [21] M.R.Setare and A.Saharian,hep-th/0002179
- [22] H.Ghafarnejad and H.Salehi.Phys.Rev.D56,1(1997)